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Modular representations of the Jordan superalgebras $D(t)$ and K_3

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Abstract

Up to the irreducible representations of the simple three-dimensional Lie algebra sl_2, we classify the unital finite-dimensional irreducible Jordan representations of the simple superalgebra $D(t)$ in the case of an algebraically closed field of characteristic $p \neq 2$. As a corollary we obtain a classification of the finite-dimensional irreducible representations of the Kaplansky superalgebra K_3 in the case of characteristic $p \neq 2$.

Keywords: Jordan superalgebra; Jordan representations

1. Introduction

Any finite-dimensional simple Jordan algebra has only a finite number of irreducible modules. In the case of Jordan superalgebras this is not true in general, as follows from the results of the present paper.

The 3-dimensional Kaplansky superalgebra K_3 and the 1-parametric family of 4-dimensional superalgebras $D(t)$ $(t \neq 0)$ are simple Jordan superalgebras for arbitrary characteristic (see [1]). If $t = 0$, then $D(0)$ contains K_3.

The superalgebra $D(t)$ has an identity element. It can be shown in the usual way that the right multiplication by the identity element for any irreducible superbimodule is either the zero

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operator, or $\frac{1}{2} \cdot \mathbf{Id}$, or $\mathbf{Id}$, where $\mathbf{Id}$ is the identity mapping. In the first case the module is trivial. The second case for the superalgebra $D(t)$ of characteristic 0 was considered by C. Martínez and E. Zelmanov in [2]. The third case for the superalgebra $D(t)$ of characteristic 0 was studied independently by C. Martínez and E. Zelmanov in [3] and by the author in [5]. As a corollary a classification of finite-dimensional irreducible representations of the Kaplansky superalgebra K_3 in the case of characteristic 0 was obtained.

In fact, in [5] the author considered a Jordan superalgebra, denoted by $D(\lambda, \mu)$, such that $D(1,t) = D(t)$. The superalgebra $D(\lambda, \mu)$ is more symmetric than $D(t)$. In this paper we study irreducible unital finite-dimensional supermodules over $D(\lambda, \mu)$ in the case of prime characteristic $p \neq 2$.

The superalgebra $D(\lambda, \mu)$ is defined as a commutative superalgebra

$$D(\lambda, \mu) = (F \cdot e_1 + F \cdot e_2) + (F \cdot x + F \cdot y),$$

with the product determined by

$$e_i^2 = e_i, \qquad e_1 \cdot e_2 = 0, \qquad e_i x = \frac{1}{2} x, \qquad e_i y = \frac{1}{2} y, \qquad xy = \lambda e_1 + \mu e_2,$$

and the grading

$$\big(D(\lambda, \mu)\big)_0 = F \cdot e_1 + F \cdot e_2, \qquad \big(D(\lambda, \mu)\big)_1 = F \cdot x + F \cdot y.$$

We will obtain the representations of $D(\lambda, \mu)$ from those of the simple 3-dimensional Lie algebra sl_2.

Let R_a be the operator of the right multiplication by a. We will also denote it by the capital letter A.

Let us denote by g, f, and h the elements of the standard basis of sl_2, such that $[g,h] = 2g$, $[f,h] = -2f$, $[g,f] = h$. The following description of the irreducible representations of sl_2 in the case of finite characteristic $p \neq 2$ was obtained by A. Rudakov and I. Shafarevich in [4]:

> The center of the universal covering algebra of sl_2 is generated by the elements $x = g^p$, $y = f^p$, $z = h^p - h$, $t = (h+1)^2 - 4gf$, these elements satisfy the relation
>
> $$z^2 - \prod_{k=0}^{p-1} \big(t - k^2\big) = 4xy,$$
>
> and any point of the variety, defined by this equation corresponds to a unique irreducible p-dimensional representation provided that $P \neq (0,0,0,k^2)$, $k \neq 0$, and the points $P = (0,0,0,k^2)$, $k \neq 0$, correspond to two irreducible representations of degrees k and $p-k$.

Note that the two irreducible representations of degrees k and $p-k$, which correspond to the point $P = (0,0,0,k^2)$, $k \neq 0$, come from characteristic 0 case.

It is easily seen that for any finite-dimensional irreducible module over sl_2 one can choose a basis $l_0, \ldots, l_n$, such that

$$l_i f = l_{i+1} \quad \text{for any } i < n,$$
$$l_i h = (\gamma - 2i) l_i,$$
$$l_0 g = \alpha l_n, \quad l_n f = \beta l_0,$$

where α, β, γ belong to F. We will call this basis *standard* and we will denote by $L(n+1, \alpha, \beta, \gamma)$ the irreducible finite-dimensional sl_2-module with the standard basis $l_0, \ldots, l_n$.

For any module over $D(\lambda, \mu)$ the operators $\frac{2}{\lambda+\mu} X \circ Y$, $\frac{2}{\lambda+\mu} X^2$ and $\frac{2}{\lambda+\mu} Y^2$ generate sl_2 as a vector space (see Proposition 4.1).

We give next some examples of unital irreducible Jordan superbimodules over the superalgebra $D(\lambda, \mu)$.

1. *$M(n+1, n+2)$ over $D(\lambda, \mu)$, where $-\frac{2\lambda}{\lambda+\mu} = n \neq p-1$ is an integer*:
$M_0 = L(n+1, 0, 0, n)$. Let $l_0, \ldots, l_n$ be a standard basis of M_0. The elements $l_0 x, l_0 y, l_1 y, l_2 y, \ldots, l_n y$ form a basis of M_1. The multiplication by the elements of the superalgebra is defined by

$$l_0 x Y = \frac{\mu - \lambda}{2} l_0,$$
$$l_{k+1} X = (n-k) l_k y \quad \text{for every } k < n,$$
$$l_k y X = -\frac{(\lambda+\mu)(k+1)}{2} l_k,$$
$$E_1 \equiv 0 \quad \text{on } M_0,$$
$$E_1 \equiv \frac{1}{2} \quad \text{on } M_1.$$

2. *$M(n+1, n)$ over $D(\lambda, \mu)$, where $-\frac{2\mu}{\lambda+\mu} = n \neq 0$ is an integer*:
In this case $M_0 = L(n+1, 0, 0, n)$. Denote by $l_0, \ldots, l_n$ a standard basis of M_0. Then M_1 has a basis formed by $l_1 x, \ldots, l_n x$,

$$l_0 X = l_n Y = 0,$$
$$l_k Y = -\frac{1}{k+1} l_{k+1} X \quad \text{for every } k < n,$$
$$E_1 \equiv 0 \quad \text{on } M_0,$$
$$E_1 \equiv \frac{1}{2} \quad \text{on } M_1.$$

Note that the given equations are enough to define the multiplication because

$$l_i x Y = -i l_{i-1} Y^2 = -i \cdot \frac{\lambda+\mu}{2} l_i,$$

by the definition of a standard basis.

3. *$M(\beta, \gamma)$, where $\beta \neq 0$*:
We take $M_0 = L(p, \frac{((\lambda+\mu)\gamma+2\mu)((\lambda+\mu)\gamma+2\lambda)}{8(\lambda+\mu)\beta}, \frac{2}{(\lambda+\mu)} \cdot \beta, \gamma)$ with a standard basis $l_0, \ldots, l_{p-1}$, then $l_0 y, l_1 y, \ldots, l_{p-1} y$ form a basis of M_1,

$$l_{p-1}Y^2 = \beta l_0,$$
$$l_0X = \frac{(\lambda+\mu)\gamma + 2\mu}{4\beta} l_{p-1}y,$$
$$l_{i+1}X = \frac{(\lambda+\mu)(\gamma-2i) - 2\lambda}{2(\lambda+\mu)} l_i y \quad \text{for every } i < p-1,$$
$$l_i yX = \frac{(\lambda+\mu)(\gamma-2i) - 2\mu}{4} l_i,$$
$$E_1 \equiv 0 \quad \text{on } M_0,$$
$$E_1 \equiv \frac{1}{2} \quad \text{on } M_1.$$

4. *$M(\alpha)$ over $D(\lambda,\mu)$, where $\lambda \neq \mu$ and $-\frac{2\mu}{\lambda+\mu}$ is not an integer*:
We define M_0 as the irreducible sl_2-module $L(p, \frac{2}{\lambda+\mu}\cdot\alpha, 0, -\frac{2\mu}{\lambda+\mu})$ with a standard basis $l_0,\dots,l_{p-1}$, the odd part M_1 has a basis formed by $l_0y, l_1y, \dots, l_{p-1}y$,

$$l_0X = \frac{2\alpha}{\lambda-\mu} l_{p-1}y,$$
$$l_{p-1}Y^2 = 0,$$
$$l_{i+1}X = -(i+1)l_i y \quad \text{for every } i < p-1,$$
$$l_i yX = -\frac{i(\lambda+\mu) + 2\mu}{2} l_i,$$
$$E_1 \equiv 0 \quad \text{on } M_0,$$
$$E_1 \equiv \frac{1}{2} \quad \text{on } M_1.$$

5. $M(l_{p-1}y = 0)$:
The even part $M_0 = L(p, \frac{2}{\lambda+\mu}\cdot\alpha, 0, -\frac{2\lambda}{\lambda+\mu})$ has a standard basis $l_0,\dots,l_{p-1}$, the elements $l_0x, l_0y, l_1y, \dots, l_{p-2}y$ form a basis of M_1,

$$l_{p-1}y = 0,$$
$$l_{i+1}X = -\frac{i(\lambda+\mu) + 2\lambda}{\lambda+\mu} l_i y \quad \text{for every } i < p-1,$$
$$l_i yX = -\frac{(\lambda+\mu)(i+1)}{2} l_i,$$
$$l_0XY = \frac{\mu-\lambda}{2} l_0,$$
$$l_0X^2 = \alpha l_{p-1},$$
$$E_1 \equiv 0 \quad \text{on } M_0,$$
$$E_1 \equiv \frac{1}{2} \quad \text{on } M_1.$$

Note that this module is irreducible iff $\alpha \neq 0$ or $\lambda \neq \mu$.

6. $M(n, \alpha, \beta, \gamma)$:
We define the even part M_0 as a vector space by taking the basis $l_0, \ldots, l_n$, $l_0[X, Y], \ldots, l_n[X, Y]$, where $l_0, \ldots, l_n$ form a standard basis of an irreducible sl_2-module $L(n+1, \frac{2}{\lambda+\mu} \cdot \alpha, \frac{2}{\lambda+\mu} \cdot \beta, \gamma)$; M_1 has a basis formed by $l_0 X, \ldots, l_n X$, $l_0 Y, \ldots, l_n Y$,

$$l_i[X, Y]E_1 = l_i[X, Y] - \mu l_i,$$
$$\left[X^2, Y\right] \equiv \frac{\lambda+\mu}{2} X,$$
$$\left[Y^2, X\right] \equiv -\frac{\lambda+\mu}{2} Y,$$
$$E_1 \equiv 0 \quad \text{on } L,$$
$$E_1 \equiv \frac{1}{2} \quad \text{on } M_1.$$

This module is irreducible iff $\gamma + 1 \neq \pm \frac{\sqrt{(\lambda-\mu)^2+16\alpha\beta}}{\lambda+\mu}$.

Note that the superalgebra $D(\lambda, \mu)$ is invariant with respect to the interchange of λ with μ and e_1 with e_2. So, if M is a Jordan module over $D(\lambda, \mu)$, then the module which is obtained from M by interchanging of λ with μ and E_1 with E_2 is also Jordan over $D(\lambda, \mu)$.

Our main result is the following:

Theorem 1.1. *Let M be a finite-dimensional irreducible unital Jordan $D(\lambda, \mu)$-module ($\lambda + \mu \neq 0$) over an algebraically closed field F of char $p \neq 2$. Then either M is isomorphic to a 1-dimensional vector space generated by an element m such that $mX = mY = 0$, $mE_1 = \frac{1}{2}m$ (and $\lambda = \mu$) or, up to the change of grading and up to the interchange of λ with μ and e_1 with e_2, there exists an irreducible nonzero $sl_2(F)$-module $L = L(n+1, \alpha, \beta, \gamma)$, such that $L \subseteq M_0$, $E_1 \equiv 0$ on L and we have the following cases*:

1. *if $L = M$, then M is isomorphic to a 1-dimensional vector space generated by an element m such that $mX = mY = mE_1 = 0$ (and $\mu = 0$)*;
2. *if $\dim L = n+1 < p$, then $\alpha = \beta = 0$ and M is isomorphic to*:

$$M(n+1, n+2) \quad \textit{if } n = -\frac{2\lambda}{\lambda+\mu} \ (\textit{as elements in } F),$$
$$M(n+1, n) \quad \textit{if } n = -\frac{2\mu}{\lambda+\mu},$$
$$M(n, 0, 0, n), \quad \textit{otherwise};$$

3. *if $\dim L = p$, $\beta \neq 0$, then M is isomorphic to either*

$$M\left(\frac{\lambda+\mu}{2} \cdot \beta, \gamma\right) \quad \textit{or}$$

$$M(p, \alpha, \beta, \gamma), \quad \textit{in the latter case } \gamma + 1 \neq \pm \frac{\sqrt{(\lambda-\mu)^2 + 16\alpha\beta}}{\lambda+\mu};$$

4. *if* $\dim L = p$, $\beta = 0$, $\alpha \neq 0$, *then* M *is isomorphic to*:

$$M\left(\frac{\lambda+\mu}{2}\cdot\alpha\right) \quad \text{if } \gamma = -\frac{2\mu}{\lambda+\mu};$$
$$M(l_{p-1}y = 0) \quad \text{if } \gamma = -\frac{2\lambda}{\lambda+\mu};$$
$$M(p,\alpha,0,\gamma), \quad \text{otherwise};$$

5. *if* $\dim L = p$, $\beta = \alpha = 0$, *then* M *is isomorphic to*:

$$M\left(\frac{\lambda+\mu}{2}\cdot\alpha\right) \quad \text{if } \gamma = -\frac{2\mu}{\lambda+\mu} \text{ and } \lambda \neq \mu;$$
$$M(l_{p-1}y = 0) \quad \text{if } \gamma = -\frac{2\lambda}{\lambda+\mu} \text{ and } \lambda \neq \mu;$$
$$M(p,0,0,\gamma), \quad \text{provided that } \gamma = -\frac{2\lambda}{\lambda+\mu},\ \gamma = -\frac{2\mu}{\lambda+\mu}.$$

We recall the main definitions. Let $\mathfrak{M}$ be a homogeneous variety of algebras. $A = A_0 \oplus A_1$ is called an $\mathfrak{M}$-superalgebra if its Grassmann envelope

$$G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$$

(where $G = G_0 \oplus G_1$ is the Grassmann algebra with the natural Z_2-grading) is an $\mathfrak{M}$-algebra.

Jordan superalgebras also can be defined by means of the identity of supercommutativity

$$a_i a_j = (-1)^{ij} a_j a_i$$

and of the following identity:

$$\begin{aligned} &R_{a_i}R_{a_j}R_{a_k} + (-1)^{ij+ik+jk}R_{a_k}R_{a_j}R_{a_i} + (-1)^{jk}R_{(a_ia_k)a_j} \\ &\quad = R_{a_i}R_{a_ja_k} + (-1)^{ij+ik+jk}R_{a_k}R_{a_ja_i} + (-1)^{ij}R_{a_j}R_{a_ia_k}, \end{aligned} \tag{1}$$

where $i, j, k, \in \{0, 1\}$, $a_i \in A_i$.

Set $[A, B]^g = AB - (-1)^{\alpha\beta}BA$, where $A : M_i \to M_{i+\alpha}$ and $B : M_i \to M_{i+\beta}$ are endomorphisms of M.

We shall need the following identity, which is satisfied in arbitrary Jordan superalgebra [1]:

$$(-1)^{jk+1}R_{(a_i,a_j,a_k)} = \big[[R_{a_i}, R_{a_k}]^g, R_{a_j}\big]^g, \tag{2}$$

where $(a, b, c) = (ab)c - a(bc)$.

A superbimodule over A is a linear space $M = M_0 \oplus M_1$ over a field F with two bilinear mappings

$$A \times M \to M, \quad M \times A \to M,$$

such that $A_iM_j + M_jA_i \subseteq M_{i+j}$, where the indexes belong to $\mathbf{Z}_2$. In what follows by a *module* we shall understand a superbimodule.

A superbimodule M over a superalgebra A is called an $\mathfrak{M}$-superbimodule if the split extension $R = M \oplus A$, where $M^2 = 0$, with the Z_2-grading given by $R_0 = M_0 \oplus A_0$, $R_1 = M_1 \oplus A_1$, is an $\mathfrak{M}$-superalgebra.

2. Basic relations

In this section we will prove some basic identities for the unital representations of the superalgebra $D(\lambda, \mu)$. We shall work with a unital module M over $D(\lambda, \mu)$ and so $E_2 = \mathbf{Id} - E_1$, where $\mathbf{Id}$ is the identity mapping. We will use the symbol E to denote the operator of right multiplication by e_1.

If we replace in (1) a_i, a_j, a_k by e, x, y, respectively, we obtain

$$YXE - EXY + (\lambda - \mu)E^2 + \frac{3\mu - \lambda}{2}E - \frac{\mu}{2}\mathbf{Id} + \frac{1}{2}[X, Y] = 0. \tag{3}$$

In the same way, we have

$$XYE - EYX - (\lambda - \mu)E^2 - \frac{3\mu - \lambda}{2}E + \frac{\mu}{2}\mathbf{Id} - \frac{1}{2}[X, Y] = 0. \tag{4}$$

By (2),

$$\left[X^2, Y\right] = \frac{\lambda + \mu}{2}X, \tag{5}$$

and

$$\left[Y^2, X\right] = -\frac{\lambda + \mu}{2}Y. \tag{6}$$

The next identity holds for every associative algebra: $[X, X \circ Y] = [X^2, Y]$. Hence,

$$[X, X \circ Y] = \frac{\lambda + \mu}{2}X. \tag{7}$$

Similarly,

$$[Y, X \circ Y] = -\frac{\lambda + \mu}{2}Y. \tag{8}$$

Replacing in (1) a_i, a_j, a_k by x, e, y, respectively, we obtain

$$XEY - YEX + \frac{1}{2}[Y, X] + (\lambda - \mu)E - (\lambda - \mu)E^2 = 0. \tag{9}$$

Substituting x, e, x for a_i, a_j, a_k in (2), we get

$$\left[X^2, E\right] = 0. \tag{10}$$

Likewise,

$$\big[Y^2, E\big] = 0. \tag{11}$$

Replacing now a_i, a_j, a_k in (1) by x, e, e, respectively, we obtain

$$XE^2 + E^2X - XE - EX + \frac{1}{4}X = 0. \tag{12}$$

Analogously,

$$YE^2 + E^2Y - YE - EY + \frac{1}{4}Y = 0. \tag{13}$$

Summing (3) and (4), we have

$$[E, X \circ Y] = 0. \tag{14}$$

Using (5)–(8), we come to

$$\big[[X, Y], X^2\big] = \big[[X, Y], Y^2\big] = \big[[X, Y], X \circ Y\big] = 0. \tag{15}$$

3. The Pierce decomposition

In the rest of the paper we assume that $\lambda + \mu \neq 0$. With respect to the idempotent $e_1 \in D(\lambda, \mu)$ the Pierce decomposition of M is

$$M = M^0 \oplus M^{\frac{1}{2}} \oplus M^1,$$

where $M^0 = \{m \in M \mid mE = 0\}$, $M^{\frac{1}{2}} = \{m \in M \mid mE = \frac{1}{2}m\}$, $M^1 = \{m \in M \mid mE = m\}$. It follows by the same method as in the case of Jordan algebras that the Pierce decomposition $D(\lambda, \mu) = D^0 + D^{\frac{1}{2}} + D^1$ with respect to e_1 satisfies the following conditions: $M^0 D^{\frac{1}{2}} + M^1 D^{\frac{1}{2}} \subseteq M^{\frac{1}{2}}$, $M^{\frac{1}{2}} D^{\frac{1}{2}} \subseteq M^0 + M^1$. From this we have $M^0X + M^0Y + M^1X + M^1Y \subseteq M^{\frac{1}{2}}$. Moreover, these conditions imply that $M^{\frac{1}{2}}$ is invariant under the even operators.

Suppose that $M^{\frac{1}{2}} = 0$. Then $E^2 = E$. Since $M^0X + M^0Y + M^1X + M^1Y \subseteq M^{\frac{1}{2}}$, it follows that $X = Y = 0$. But then by (3), $\frac{\lambda+\mu}{2}E - \frac{\mu}{2}\mathbf{Id} = 0$. Thus $E = \frac{\mu}{\lambda+\mu}\mathbf{Id}$. Hence $\frac{\mu}{\lambda+\mu} = 0$ or $\frac{\mu}{\lambda+\mu} = 1$.

In the first case we have $\mu = 0$, the irreducible module M is a 1-dimensional vector space generated by an element m such that $mX = mY = mE = 0$.

In the second case: $\lambda = 0$, the module M is a 1-dimensional vector space generated by an element m such that $mX = mY = 0$, $mE = m$.

Note that, in the second case interchanging e_1 with e_2 and λ with μ we come to the first case.

In what follows we assume that $M^{\frac{1}{2}} \neq 0$. Since $M^{\frac{1}{2}}$ is invariant under the even operators and $M^{\frac{1}{2}} D^{\frac{1}{2}} \subseteq M^0 + M^1$, it follows that $M^{\frac{1}{2}} = M_0$ or $M^{\frac{1}{2}} = M_1$ for any irreducible module M.

4. The subalgebra sl_2 of the algebra of even operators

Proposition 4.1. *The operators* $\frac{2}{\lambda+\mu}X\circ Y$, $\frac{2}{\lambda+\mu}X^2$ *and* $\frac{2}{\lambda+\mu}Y^2$ *generate the simple Lie algebra* sl_2 *as a vector space.*

Proof. Using (7), we obtain

$$\left[\frac{2}{\lambda+\mu}X^2, \frac{2}{\lambda+\mu}X\circ Y\right] = \frac{4}{(\lambda+\mu)^2}\cdot(\lambda+\mu)X^2 = 2\cdot\frac{2}{\lambda+\mu}X^2.$$

By (8),

$$\left[\frac{2}{\lambda+\mu}Y^2, \frac{2}{\lambda+\mu}X\circ Y\right] = -\frac{4}{(\lambda+\mu)^2}\cdot(\lambda+\mu)Y^2 = -2\cdot\frac{2}{\lambda+\mu}Y^2.$$

Using (5), we have

$$\left[\frac{2}{\lambda+\mu}X^2, \frac{2}{\lambda+\mu}Y^2\right] = \frac{4}{(\lambda+\mu)^2}\cdot\frac{\lambda+\mu}{2}X\circ Y = \frac{2}{\lambda+\mu}X\circ Y.$$

This completes the proof. □

Without loss of generality we can assume that $M^{\frac{1}{2}} = M_1$. The even part M_0 contains an irreducible sl_2-module L. Observe that in view of (10), (11), (14), L may be chosen such that E is scalar on L. If $M_0 \neq 0$, then the above L can be chosen nonzero. If $M_0 = 0$, then $M = M_1 = M^{\frac{1}{2}}$ is a 1-dimensional vector space with a generator m such that $mX = mY = 0$, $mE = \frac{1}{2}m$. From (9) it follows that in this case we necessarily have $\lambda = \mu$.

For the rest of the paper we assume that $L \neq 0$.

By (4), $LXYE \subseteq L + L[X,Y]$. Similarly, $LYXE \subseteq L + L[X,Y]$. By (6) and (5), $LXY^2 + LYX^2 \subseteq LX + LY$. Finally note that $LXYX \subseteq L(X\circ Y)X + LYX^2 \subseteq LX + LY$ and, by a similar argument, $LYXY \subseteq LX + LY$. Hence $M_0 = L + L[X,Y]$, $M_1 = LX + LY$.

If $E = 1$ on L then interchanging e_1 and e_2, λ and μ we obtain the case $E = 0$ on L.

Now suppose that $E = 0$ on L. Note that, by (15), $l \mapsto l[X,Y]$ is a Lie module homomorphism of L into $L[X,Y]$. The module L is irreducible, hence $L[X,Y] = 0$ or $L[X,Y] \cong L$.

Also note that from (7) it follows that the operator X increases the weight by 1. More precisely, if $mX\circ Y = \frac{\lambda+\mu}{2}\gamma m$, then $mX(X\circ Y) = \frac{\lambda+\mu}{2}(\gamma+1)mX$. Similarly, $mY(X\circ Y) = \frac{\lambda+\mu}{2}(\gamma-1)mY$.

5. The case $L[X,Y] \subseteq L$

Suppose that $L[X,Y] \subseteq L$. Then $M_0 = L$, $E = 0$ on M_0 and, by (3), $[X,Y] = \mu\cdot\mathbf{Id}$ on M_0. Let $l_0, l_1, \dots, l_n$ be a basis of $M_0 = L$ such that l_i is a vector corresponding to the weight $\gamma - 2i$ and $l_iY^2 = \frac{\lambda+\mu}{2}l_{i+1}$ for any $i < n$, $l_nY^2 = \beta l_0$. Then

$$l_iXY = \frac{(\lambda+\mu)(\gamma-2i)+2\mu}{4}l_i \quad \text{for every } i \tag{16}$$

and

$$l_i YX = \frac{(\lambda+\mu)(\gamma-2i)-2\mu}{4} l_i \quad \text{for every } i. \tag{17}$$

Multiplying both sides of (16) by Y and using (6), we obtain

$$l_{i+1}X = \frac{(\lambda+\mu)(\gamma-2i-2)+2\mu}{2(\lambda+\mu)} l_i y \quad \text{for } i < n.$$

Similarly we have

$$\beta l_0 X = \frac{(\lambda+\mu)(\gamma-2n-2)+2\mu}{4} l_n y.$$

If $\beta \neq 0$, then we obtain the following module:

M_0 is the sl_2-module with the above basis $l_0, \dots, l_{p-1}$. The elements $l_0 y, l_1 y, \dots, l_{p-1} y$ form a basis of M_1. The multiplication by elements of $D(\lambda, \mu)$ is defined by

$$l_0 X = \frac{(\mu+\lambda)\gamma+2\mu}{4\beta} l_{p-1} Y,$$
$$l_{i+1}X = \frac{(\lambda+\mu)(\gamma-2i)-2\lambda}{2(\lambda+\mu)} l_i y \quad \text{for every } \ i < p-1,$$
$$l_i yX = \frac{(\lambda+\mu)(\gamma-2i)-2\mu}{4} l_i,$$
$$E \equiv 0 \quad \text{on } M_0, \qquad E \equiv \frac{1}{2} \quad \text{on } M_1.$$

If $\beta = 0$, then $(\lambda+\mu)(\gamma-2n-2)+2\mu = 0$ or $l_n y = 0$.
Note that, by (17), the following conditions are equivalent:

1. $l_n y = 0$;
2. $\gamma - 2n = \frac{2\mu}{\lambda+\mu}$.

First assume that $l_n y \neq 0$. Then $\gamma - 2n = \frac{2\lambda}{\lambda+\mu}$. By the above,

$$l_0 YX = \frac{(\lambda+\mu)n+\lambda-\mu}{2} l_0.$$

Multiplying both sides by X and using (5), we obtain

$$\frac{(\lambda+\mu)n+2\lambda}{2} l_0 X = \alpha l_n Y,$$

where $\alpha \in F$ is determined by the Lie module L $(l_0 X^2 = \alpha l_n)$.
If $\alpha \neq 0$, then $n = p-1$, $\gamma = -\frac{2\mu}{\lambda+\mu}$ and $\alpha l_{p-1} Y = \frac{\lambda-\mu}{2} l_0 X$. Hence, $\lambda \neq \mu$ and $l_0 X = \frac{2\alpha}{\lambda-\mu} l_{p-1} Y$. We have the following module:

M_0 is the sl_2-module with the above basis $l_0, \dots, l_{p-1}$. The odd part M_1 has a basis formed by $l_0 y, l_1 y, l_2 y, \dots, l_{p-1} y$. The multiplication by elements of $D(\lambda, \mu)$ is defined by

$$l_0 X = \frac{2\alpha}{\lambda - \mu} l_{p-1} Y,$$
$$l_{i+1} X = -(i+1) l_i y \quad \text{for every } i < p-1,$$
$$l_i y X = -\frac{i(\lambda + \mu) + 2\mu}{2} l_i \quad \text{for any } i,$$
$$E \equiv 0 \quad \text{on } M_0, \qquad E \equiv \frac{1}{2} \quad \text{on } M_1.$$

Suppose now that $\alpha = 0$. Then $(\lambda + \mu)n + 2\lambda = 0$ or $l_0 X = 0$.

In the first case $n < p - 1$. Indeed, if $n = p - 1$ then $\lambda = \mu$ and $\gamma - 2n = \frac{2\lambda}{\lambda+\mu} = \frac{2\mu}{\lambda+\mu}$. Hence, $l_n Y = 0$ by above, a contradiction.

Since $n < p - 1$, it follows that $\gamma = n = -\frac{2\lambda}{\lambda+\mu}$ and $\lambda \neq \mu$. By (16), $l_0 X Y = \frac{\mu-\lambda}{2} l_0$. Therefore $l_0 X \neq 0$. Since $l_0 x, l_0 y, l_1 y, \dots, l_n y$ are vectors corresponding to the weights $n+1, n-1, \dots, -n-1$, it follows that these vectors are linearly independent. We have the following module:

M_0 is the irreducible sl_2-module with the basis $l_0, \dots, l_n$. The following elements form a basis of M_1: $l_0 x, l_0 y, l_1 y, \dots, l_n y$. The multiplication by the elements of the superalgebra is defined by

$$l_0 X^2 = l_n Y^2 = 0, \qquad l_0 x Y = \frac{\mu - \lambda}{2} l_0,$$
$$l_{i+1} X = (n - i) l_i y \quad \text{for every } i < n,$$
$$l_i y X = -\frac{(\lambda + \mu)(i+1)}{2} l_i,$$
$$E \equiv 0 \quad \text{on } M_0,$$
$$E \equiv \frac{1}{2} \quad \text{on } M_1.$$

Now let $l_0 X = 0$. Then by (16), $(\lambda + \mu)\gamma + 2\mu = 0$. Hence by the above, $n + 1 = 0$. Since $l_{p-1} Y \neq 0$, it follows that $\lambda \neq \mu$. We have the following module:

The even part M_0 is the irreducible sl_2-module with the basis $l_0, \dots, l_{p-1}$. The elements $l_0 y, l_1 y, \dots, l_{p-1} y$ form a basis of M_1. The multiplication by the elements of the superalgebra is defined by

$$l_0 x = l_{p-1} y Y = 0,$$
$$l_{i+1} X = -(i+1) l_i y \quad \text{for every } i < p-1,$$
$$l_i y X = -\frac{(\lambda + \mu) i + 2\mu}{2} l_i,$$
$$E_1 \equiv 0 \quad \text{on } M_0,$$
$$E_1 \equiv \frac{1}{2} \quad \text{on } M_1.$$

So, this module is $M(\alpha)$ for $\alpha = 0$.

Now we consider the case $l_nY = 0$. Then, by the above, $\gamma - 2n = \frac{2\mu}{\lambda+\mu}$. By (17), $l_0YX = \frac{n(\lambda+\mu)}{2}l_0$. Multiplying both sides by X and using (5), we obtain $\frac{(n+1)(\lambda+\mu)}{2}l_0X = 0$. Hence, $n = p-1$ or $l_0X = 0$.

If $l_0X \neq 0$, then $\gamma = -\frac{2\lambda}{\lambda+\mu}$ and we obtain the following module:

M_0 is the sl_2-module with the above basis $l_0, \dots, l_{p-1}$. The odd part M_1 has a basis formed by $l_0x, l_0y, l_1y, l_2y, \dots, l_{p-2}y$. The multiplication by the elements of $D(\lambda, \mu)$ is defined by

$$l_{p-1}Y = 0,$$

$$l_{i+1}X = -\frac{i(\lambda+\mu)+2\lambda}{\lambda+\mu}l_iy \quad \text{for every } i < p-1,$$

$$l_0xY = \frac{\mu-\lambda}{2}l_0,$$

$$l_iyX = -\frac{(i+1)(\lambda+\mu)}{2}l_i \quad \text{for any } \ i,$$

$$l_0X^2 = \alpha l_{p-1},$$

$$E \equiv 0 \quad \text{on } M_0, \qquad E \equiv \frac{1}{2} \quad \text{on } M_1.$$

Note that $\lambda \neq \mu$ or $\alpha \neq 0$ in this case.

Finally, let $l_0X = 0$. Then by (16), $\gamma = -\frac{2\mu}{\lambda+\mu}$ and $n = \gamma$. We obtain the following module:

M_0 is the irreducible sl_2-module with the basis $l_0, \dots, l_n$, the odd part M_1 has a basis formed by $l_1x, \dots, l_nx$,

$$l_0X = l_nY = 0,$$

$$l_iY = -\frac{1}{i+1}l_{i+1}X \quad \text{for every } i < n,$$

$$E_1 \equiv 0 \quad \text{on } M_0,$$

$$E_1 \equiv \frac{1}{2} \quad \text{on } M_1.$$

Note that $n \neq 0$, by $M_1 \neq 0$.

6. The case $L[X, Y] \nsubseteq L$

Suppose that $L = L(n+1, \frac{2}{\lambda+\mu}\cdot\alpha, \frac{2}{\lambda+\mu}\cdot\beta, \gamma)$. By the results of Section 4, the elements $l_0, \dots, l_n, l_0[X, Y], \dots, l_n[X, Y]$ form a basis of M_0, where $l_0, \dots, l_n$ is a standard basis of L. For any i, $l_iX \neq 0, l_iY \neq 0$, which follows from $L[X, Y] \nsubseteq L$. Moreover, l_iX is a vector corresponding to the weight $\gamma - 2i + 1$ and l_iY is a vector corresponding to the weight $\gamma - 2i - 1$. This implies that only l_iX and $l_{i-1}Y$ can be linearly dependent. Suppose that there exists i such that $l_iX = \delta l_{i-1}Y$, $\delta \in F$. Then $l_iXY = \delta l_{i-1}Y^2 \in L$, hence $l_i[X, Y] = l_i(2XY - X \circ Y) \in L$, a contradiction.

Thus, we obtain that the elements $l_0X, \ldots, l_nX, l_0Y, \ldots, l_nY$ form a basis of M_1. The multiplication by the elements of $D(\lambda, \mu)$ is defined by the construction of L, by (5), (6) and by

$$\begin{gathered} l_i[X,Y]E = l_i[X,Y] - \mu l_i, \quad \text{by (4)–(3)}, \\ E \equiv 0 \quad \text{on } L, \\ E \equiv \frac{1}{2} \quad \text{on } M_1. \end{gathered}$$

We shall prove that this module is irreducible if and only if $\gamma + 1 \neq \pm\frac{\sqrt{(\lambda-\mu)^2+16\alpha\beta}}{\lambda+\mu}$.

It is easily seen that if there exists a proper submodule N, then its even part is not zero and $m = l_0[X,Y] - \mu l_0$ belongs to N. In this case mXY also belongs to N.

By (5),

$$\begin{aligned} mXY &= l_0(X \circ Y - 2YX)XY - \mu l_0 XY \\ &= l_0(X \circ Y)XY - 2l_0\bigl(YX^2\bigr)Y - \mu l_0 XY \\ &= \left(\frac{\lambda+\mu}{2}\gamma - \mu\right) l_0 XY + (\lambda+\mu) l_0 XY - 2\alpha\beta l_0 \\ &= \frac{(\lambda+\mu)\gamma + 2\lambda}{4}\bigl(l_0(X \circ Y) + l_0[X,Y]\bigr) - 2\alpha\beta l_0 \\ &= \frac{(\lambda+\mu)\gamma + 2\lambda}{4} l_0[X,Y] + \left(\frac{(\lambda+\mu)^2\gamma^2 + 2\lambda(\lambda+\mu)\gamma}{8} - 2\alpha\beta\right) l_0. \end{aligned}$$

Subtracting $\frac{(\lambda+\mu)\gamma+2\lambda}{4} m$ from mXY we see that the element

$$\left(\frac{(\lambda+\mu)^2}{4}\gamma^2 + \frac{(\lambda+\mu)^2}{2}\gamma - 4\alpha\beta + \lambda\mu\right) l_0$$

is contained in N. Hence,

$$\gamma^2 + 2\gamma + \frac{4(-4\alpha\beta + \lambda\mu)}{(\lambda+\mu)^2} = 0.$$

This gives $\gamma + 1 = \pm\frac{\sqrt{(\lambda-\mu)^2+16\alpha\beta}}{\lambda+\mu}$. It is easy to check that in this case the element m generates a proper submodule.

7. The obtained modules are Jordan

Note that a unital module over $D(\lambda, \mu)$ is Jordan if and only if the identities (3), (5), (9), (10), (12) are satisfied as well as the following identities

$$\begin{gathered} 2E^3 - 3E^2 + E = 0, \\ 2EXE - EX - XE + \frac{1}{2}X = 0, \end{gathered}$$

and the identities obtained from all of them replacing X by Y and Y by $-X$.

First we prove that $M(n+1,n+2)$, $M(n+1,n)$, $M(\alpha)$, $M(\beta,\gamma)$, $M(l_{p-1}y=0)$ are Jordan.

Since $E_1 \equiv 0$ on M_0 and $E_1 \equiv \frac{1}{2}$ on M_1, it follows that it suffices to prove that $[X,Y]=\mu$ on M_0 and $[X,Y]=\frac{\lambda-\mu}{2}$ on M_1.

By the definition of these modules,

$$l_i[X,Y]=l_iX\circ Y-2l_iYX=\left[\frac{\lambda+\mu}{2}(\gamma-2i)-\frac{(\lambda+\mu)(\gamma-2i)-2\mu}{2}\right]l_i=\mu l_i.$$

For $i<n\leqslant p-1$ we have

$$\begin{aligned}
l_iy[X,Y]&=\frac{(\lambda+\mu)(\gamma-2i)-2\mu}{4}l_iy-\frac{\lambda+\mu}{2}l_{i+1}X\\
&=\frac{(\lambda+\mu)(\gamma-2i)-2\mu}{4}l_iy-\frac{\lambda+\mu}{2}\cdot\frac{(\lambda+\mu)(\gamma-2i)-2\lambda}{2(\lambda+\mu)}l_iy\\
&=\left(-\frac{\mu}{2}+\frac{\lambda}{2}\right)l_iy.
\end{aligned}$$

In the case of the modules $M(\beta,\gamma)$ we obtain

$$l_{p-1}y[X,Y]=\frac{(\lambda+\mu)(\gamma+2)-2\mu}{4}l_{p-1}y-\beta l_0X=\frac{\lambda-\mu}{2}l_{p-1}y.$$

For the modules $M(n+1,n+2)$ and $M(\alpha)$ we have

$$l_ny[X,Y]=-\frac{(\lambda+\mu)(-\frac{2\lambda}{\lambda+\mu}+1)}{2}l_ny=\frac{\lambda-\mu}{2}l_ny$$

and

$$l_{p-1}y[X,Y]=-\frac{-(\lambda+\mu)+2\mu}{2}l_{p-1}y=\frac{\lambda-\mu}{2}l_{p-1}y.$$

In the case of the modules $M(n+1,n+2)$ and $M(l_{p-1}y=0)$ we obtain $l_0x[X,Y]=-\frac{\mu-\lambda}{2}l_0x$.

So, we proved that $M(n+1,n)$, $M(n+1,n+2)$, $M(\alpha)$, $M(\beta,\gamma)$ and $M(l_{p-1}y=0)$ are Jordan.

Now we show that the modules $M(n,\alpha,\beta,\gamma)$ are also Jordan. From the definition of $M(n,\alpha,\beta,\gamma)$ it follows that it suffices to prove that

$$\begin{gathered}
-l_iYXE=\frac{1}{2}l_i[X,Y]-\frac{\mu}{2}l_i,\\
l_i[X,Y]\left(YXE-EXY+\frac{\lambda+\mu}{2}E-\frac{\mu}{2}\cdot\mathbf{Id}+\frac{1}{2}[X,Y]\right)=0
\end{gathered}$$

and

$$XEY-YEX+\frac{1}{2}[Y,X]+\frac{\lambda-\mu}{4}\cdot\mathbf{Id}=0\quad\text{on } M_1.$$

Since $l_i(X \circ Y)E = 0$ and $l_i[X, Y]E = l_i[X, Y] - \mu l_i$, it follows that

$$l_i XYE = -l_i YXE = \frac{1}{2}\bigl(l_i[X, Y] - \mu l_i\bigr).$$

Now, by (6), we have

$$\begin{aligned}
&\bigl(l_i[X, Y] - \mu l_i\bigr)\left(YXE - EXY + \frac{\lambda+\mu}{2}E - \frac{\mu}{2}\cdot \mathbf{Id} + \frac{1}{2}[X, Y]\right)\\
&\quad = \bigl(l_i[X, Y] - \mu l_i\bigr)\left(YXE - XY + \frac{\lambda}{2}\cdot \mathbf{Id} + \frac{1}{2}[X, Y]\right)\\
&\quad = \bigl(l_i[X, Y] - \mu l_i\bigr)\left(YXE - \frac{1}{2}X \circ Y + \frac{\lambda}{2}\cdot \mathbf{Id}\right)\\
&\quad = \bigl(l_i[X, Y] - \mu l_i\bigr)\left(YXE - \left(\frac{\lambda+\mu}{4}(\gamma - 2i) - \frac{\lambda}{2}\right)\cdot \mathbf{Id}\right)\\
&\quad = l_i\bigl(XY^2XE - YXYXE\bigr) - \left(\frac{\lambda+\mu}{4}(\gamma - 2i) - \frac{\lambda}{2}\right)\bigl(l_i[X, Y] - \mu l_i\bigr) + \frac{\mu}{2}\bigl(l_i[X, Y] - \mu l_i\bigr)\\
&\quad = l_i\left(2\left(\frac{\lambda+\mu}{2}YXE + Y^2X^2E\right) - (X \circ Y)YXE\right) - \frac{\lambda+\mu}{2}\cdot\left(\frac{\gamma - 2i}{2} - 1\right)\bigl(l_i[X, Y] - \mu l_i\bigr)\\
&\quad = (\lambda+\mu)l_i YXE - \frac{\lambda+\mu}{2}(\gamma - 2i)l_i YXE - \frac{\lambda+\mu}{2}\cdot\left(\frac{\gamma - 2i}{2} - 1\right)\bigl(l_i[X, Y] - \mu l_i\bigr)\\
&\quad = (\lambda+\mu)\left(-\frac{\gamma - 2i}{2} + 1\right)l_i YXE + (\lambda+\mu)\left(\frac{\gamma - 2i}{2} - 1\right)l_i YXE = 0.
\end{aligned}$$

Finally, by (6)

$$\begin{aligned}
&l_i Y\left(XEY - YEX + \frac{1}{2}[Y, X] + \frac{\lambda-\mu}{4}\cdot \mathbf{Id}\right)\\
&\quad = l_i YXEY + \frac{1}{2}l_i\bigl(Y^2X - YXY\bigr) + \frac{\lambda-\mu}{4}l_i Y\\
&\quad = \frac{\mu}{2}l_i Y - \frac{1}{2}l_i[X, Y]Y + \frac{1}{2}\left(-\frac{\lambda+\mu}{2}l_i Y + l_i XY^2 - l_i YXY\right) + \frac{\lambda-\mu}{4}l_i Y = 0
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&l_i X\left(XEY - YEX + \frac{1}{2}[Y, X] + \frac{\lambda-\mu}{4}\cdot \mathbf{Id}\right)\\
&\quad = \frac{1}{2}\bigl(\mu l_i X - l_i[X, Y]X\bigr) + \frac{1}{2}l_i\bigl(XYX - X^2Y\bigr) + \frac{\lambda-\mu}{4}l_i X = 0.
\end{aligned}$$

Therefore the modules $M(n, \alpha, \beta, \gamma)$ are Jordan.

8. Bimodules over the superalgebra $D(t)$ and over the Kaplansky superalgebra

The superalgebra $D(t)$ is defined as a commutative superalgebra

$$D(t) = (F \cdot e_1 + F \cdot e_2) + (F \cdot x + F \cdot y),$$

with the product

$$e_i^2 = e_i, \qquad e_1 \cdot e_2 = 0, \qquad e_i x = \frac{1}{2}x, \qquad e_i y = \frac{1}{2}y, \qquad xy = e_1 + te_2,$$

and the grading

$$\big(D(t)\big)_0 = F \cdot e_1 + F \cdot e_2, \qquad \big(D(t)\big)_1 = F \cdot x + F \cdot y.$$

If $t = -1$, the superalgebra $D(-1)$ is isomorphic to the matrix superalgebra $M_{1,1}^{(+)}(F)$. The superalgebra $D(t)$ is simple if $t \neq 0$.

Note that $D(1,t) = D(t)$, consequently from Theorem 1.1 it is not hard to obtain the classification of the irreducible unital finite-dimensional representations of $D(t)$ $(t \neq -1)$ in the characteristic $p \neq 2$ case.

The Kaplansky superalgebra K_3 is defined as a commutative superalgebra $K_3 = Fe + (Fx + Fy)$, with the product $e^2 = e,\ ex = \frac{1}{2}x,\ ey = \frac{1}{2}y,\ xy = e$, with $(K_3)_0 = Fe,\ (K_3)_1 = Fx + Fy$.

The superalgebra $D(1,0) = D(0)$ contains the subalgebra with the basis e_1, x, y. This subalgebra is isomorphic to K_3. If M is a unital irreducible Jordan module over $D(1,0)$ then M is a Jordan module over K_3. Let N be a submodule of the K_3-module M. Then $n(e_1+e_2) = n$, hence $ne_2 \in N$ for every $n \in N$. Thus N is a submodule of the $D(1,0)$-module M. Consequently, the K_3-module M is irreducible.

The converse is also true. If M is an irreducible K_3-module then we can define the action of E_2 by $\mathbf{Id} - E_1$. Then M becomes a unital irreducible $D(1,0)$-module.

Hence from Theorem 1.1 we obtain the following classification of all irreducible modules over the Kaplansky superalgebra:

Theorem 8.1. *Every finite-dimensional irreducible Jordan superbimodule over the Kaplansky superalgebra K_3 in the case of characteristic $p \neq 2$, up to the change of grading, is isomorphic to one of the following modules*:

$\tilde{M}(1,2) \cong Reg\,K_3,\ M(p-1,p),\ \tilde{M}(p-1,p-2),\ M(l_{p-1}y=0),\ \tilde{M}(l_{p-1}y=0),\ M(\alpha),\ \tilde{M}(\alpha),\ M(n,\alpha,\beta,\gamma)$, *where* $\gamma+1 \neq \pm\sqrt{1+16\alpha\beta}$, $M(\beta,\gamma),\ \tilde{M}(\beta,\gamma)$, *where* $\tilde{M}$ *is obtained from* M *by interchanging* λ *with* μ *and* e_1 *with* e_2.

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